

FINITE MORSE INDEX SOLUTIONS AND ASYMPTOTICS OF WEIGHTED NONLINEAR ELLIPTIC EQUATIONS*

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ABSTRACT. By introducing a suitable setting, we study the behavior of finite Morse index solutions of the equation

$$(1) \quad -\operatorname{div}(|x|^\theta \nabla v) = |x|^l |v|^{p-1} v \quad \text{in } \Omega \subset \mathbb{R}^N \quad (N \geq 2),$$

where $p > 1$, $\theta, l \in \mathbb{R}^1$ with $N + \theta > 2$, $l - \theta > -2$, and Ω is a bounded or unbounded domain. Through a suitable transformation of the form $v(x) = |x|^\sigma u(x)$, equation (1) can be rewritten as a nonlinear Schrödinger equation with Hardy potential

$$(2) \quad -\Delta u = |x|^\alpha |u|^{p-1} u + \frac{\ell}{|x|^2} u \quad \text{in } \Omega \subset \mathbb{R}^N \quad (N \geq 2),$$

where $p > 1$, $\alpha \in (-\infty, \infty)$ and $\ell \in (-\infty, (N-2)^2/4)$.

We show that under our chosen setting for the finite Morse index theory of (1), the stability of a solution to (1) is unchanged under various natural transformations. This enables us to reveal two critical values of the exponent p in (1) that divide the behavior of finite Morse index solutions of (1), which in turn yields two critical powers for (2) through the transformation. The latter appear difficult to obtain by working directly with (2).

1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the properties of finite Morse index solutions to the following weighted nonlinear elliptic equation

$$(1.1) \quad -\operatorname{div}(|x|^\theta \nabla v) = |x|^l |v|^{p-1} v \quad \text{in } \Omega \subset \mathbb{R}^N \quad (N \geq 2),$$

where $p > 1$, $\theta, l \in \mathbb{R}^1$, and Ω is a bounded or unbounded domain. We are particularly interested in the cases that Ω is a punctured ball $B_R(0) \setminus \{0\}$, an exterior domain $\mathbb{R}^N \setminus B_R$, or the entire space \mathbb{R}^N . Here and throughout this paper, we use $B_r(x)$ to denote the open ball in \mathbb{R}^N centered at x with radius r . We also write $B_r = B_r(0)$.

An interesting classification of finite Morse index solutions to this equation in the case $\Omega = \mathbb{R}^N$ (or $\mathbb{R}^N \setminus \{0\}$) and $\theta = l = 0$ was given by Farina [14] recently. More recently such

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solutions in the case $\theta = 0$ and $l > -2$ were considered in [7, 25]. Other recent related research on finite Morse index solutions can be found in [4, 5, 6, 9, 11, 12, 13], where more references are given.

Our interest in the general case of (1.1) was partly motivated by research on the following nonlinear Schrödinger equation with Hardy potential,

$$(P) \quad -\Delta u = |x|^\alpha |u|^{p-1} u + \frac{\ell}{|x|^2} u \text{ in } \Omega \subset \mathbb{R}^N \ (N \geq 2),$$

where $p > 1$, $\alpha \in (-\infty, \infty)$, $\ell \in (-\infty, (N-2)^2/4)$. Equations of this type (with $N \geq 3$) arise in the study of nonlinear Schrödinger equations when the field presents a (possible) singularity at the origin and have attracted extensive studies in the past three decades; see, for example, [1, 8, 15, 21, 22, 23, 24] and the references therein.

If we define

$$(1.2) \quad v(x) = |x|^\sigma u(x), \quad \sigma = \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \ell},$$

then (P) is reduced to (1.1) with $\theta = -2\sigma$ and $l = \alpha - \sigma(p+1)$. Thus (P) can be reduced to (1.1), and vice versa.

It should be noted that when $\theta \neq 0$, the term $|x|^\theta$ in (1.1) gives rise to a singularity (or degeneracy) at $x = 0$ for the elliptic operator $\operatorname{div}(|x|^\theta \nabla v)$, and the notion of Morse index for solutions of (1.1) need to be formulated with great care in order to make it consistent and useful. All the previous work on finite Morse index solutions that we are aware of dealt with elliptic equations with a uniformly elliptic operator. Therefore one might think that the form (P) is more natural to use than its equivalent equation in the form of (1.1). Our investigation here, however, suggests the opposite.

In this paper, we define the finite Morse index for (1.1) in an appropriate setting such that the stability of a solution to (1.1) is unchanged under several natural transformations. This allows us to refine the calculations in [7] to reveal two critical values of p for (1.1) that divide the behavior of finite Morse index solutions to (1.1), and through the transformation (1.2), we obtain two critical powers for (P). As will become clear below, the critical powers for (P) can be expressed by relatively simple formulas in parameters appearing in its equivalent form (1.1), but the formulas become very complicated in terms of the parameters of (P) itself, which makes them difficult to obtain by working on (P) directly. In a recent work [18], the methods of [14] and [7] are further developed and applied to (P), but the calculations turn out to be tedious in terms of the parameters appearing in (P), and the authors have not found the optimal critical power for p in the general case.

To motivate some of the questions we investigate, and to get a taste of how (1.1) may be more natural to work with than (P), we first recall two classical results of Bidaut-Véron and Véron [1] (see Theorems 3.2, 3.3, 3.4 and Remark 3.2 in [1]).

Theorem A. *Assume that $p \in (1, \infty) \setminus \{\frac{N+2+2\alpha}{N-2}\}$, $\ell < \frac{2+\alpha}{p-1} \left(N - 2 - \frac{(2+\alpha)}{p-1} \right)$ and u is a positive solution of (P) (with $N \geq 3$) in $B_R \setminus \{0\}$ such that for some positive constant C ,*

$$|x|^{\frac{2+\alpha}{p-1}} u(x) \leq C \quad \text{in } B_R \setminus \{0\}.$$

Then we have the following:

(i) *either there exists $\eta > 0$ such that*

$$\lim_{x \rightarrow 0} u(x) |x|^{\frac{(N-2-\sqrt{(N-2)^2-4\ell})}{2}} = \eta,$$

(ii) *or there exists a positive solution ω of*

$$\Delta_{S^{N-1}} \omega - \left[\frac{2+\alpha}{p-1} \left(N - 2 - \frac{2+\alpha}{p-1} \right) - \ell \right] \omega + \omega^p = 0$$

on S^{N-1} such that

$$\lim_{r \rightarrow 0} u(r\zeta) r^{\frac{2+\alpha}{p-1}} = \omega(\zeta)$$

in the $C^k(S^{N-1})$ -topology for any $k \in \mathbb{N}$.

Theorem B. *Assume that $p \in (1, \infty) \setminus \{\frac{N+2+2\alpha}{N-2}\}$, $\ell < \frac{2+\alpha}{p-1} \left(N - 2 - \frac{2+\alpha}{p-1} \right)$ and u is a positive solution of (P) (with $N \geq 3$) in $\mathbb{R}^N \setminus B_R$ such that for some positive constant C ,*

$$|x|^{\frac{2+\alpha}{p-1}} u(x) \leq C \quad \text{in } \mathbb{R}^N \setminus B_R.$$

Then we have the following:

(i) *either there exists $\eta > 0$ such that*

$$\lim_{|x| \rightarrow \infty} u(x) |x|^{\frac{(N-2+\sqrt{(N-2)^2-4\ell})}{2}} = \eta,$$

(ii) *or there exists a positive solution ω of*

$$\Delta_{S^{N-1}} \omega - \left[\frac{2+\alpha}{p-1} \left(N - 2 - \frac{2+\alpha}{p-1} \right) - \ell \right] \omega + \omega^p = 0$$

on S^{N-1} such that

$$\lim_{r \rightarrow \infty} u(r\zeta) r^{\frac{2+\alpha}{p-1}} = \omega(\zeta)$$

in the $C^k(S^{N-1})$ -topology for any $k \in \mathbb{N}$.

If $1 < p < \frac{N+2}{N-2}$, then the estimate $|x|^{\frac{2+\alpha}{p-1}} u(x) \leq C$ in Theorems A and B automatically holds for arbitrary α and ℓ ; see Theorem 6.3 in [1] (for the special case $\ell = 0$, this was first proved in [16]). The proof for this fact is based on some useful integral estimates obtained from the Bochner-Lichnerowicz-Weitzenböck formula in \mathbb{R}^N .

For the case $\ell = 0$, it was shown in [7] that such estimate continues to hold for a larger range of p provided that the solution has finite Morse index. It would be interesting to see what happens for $\ell \neq 0$. This question will be answered as a consequence of some general results in this paper for (1.1).

Let v be a positive solution of (1.1). If we define $r = |x|$, $\zeta = \frac{x}{|x|}$ and

$$z(t, \zeta) = r^{\frac{2+l-\theta}{p-1}} v(r\zeta), \quad t = \ln r,$$

then $z(t, \zeta)$ satisfies the equation

$$\begin{aligned} z_{tt} + \left(N + \theta - 2 - \frac{2(2+l-\theta)}{p-1} \right) z_t + \Delta_{S^{N-1}} z \\ - \frac{2+l-\theta}{p-1} \left[N + \theta - 2 - \frac{2+l-\theta}{p-1} \right] z + |z|^{p-1} z = 0. \end{aligned}$$

One easily sees that the arguments in the proof of Theorems 3.2 and 3.3 in [1] still work for the above equation provided that

$$\frac{2+l-\theta}{p-1} \left[N + \theta - 2 - \frac{2+l-\theta}{p-1} \right] > 0,$$

which is satisfied if

$$(1.3) \quad N + \theta > 2, \quad l - \theta > -2 \quad \text{and} \quad p > \frac{N+l}{N+\theta-2}.$$

Therefore, the proof of Theorems 3.2 and 3.3 in [1] yields the following result for (1.1) (note that $N = 2$ is allowed here).

Theorem 1.1. *Assume that (1.3) holds, $p \neq \frac{N+2+2l-\theta}{N+\theta-2}$, and v is a positive solution of (1.1) in $B_R \setminus \{0\}$ such that for some positive constant C ,*

$$|x|^{\frac{2+l-\theta}{p-1}} v(x) \leq C \quad \text{in } B_R \setminus \{0\}.$$

Then either $x = 0$ is a removable singularity or it is a nonremovable singularity and

$$r^{\frac{2+l-\theta}{p-1}} v(r\zeta) \rightarrow \varpi(\zeta) \quad \text{as } r \rightarrow 0 \text{ uniformly in } \zeta \in S^{N-1},$$

where ϖ is a positive solution of

$$(1.4) \quad \Delta_{S^{N-1}} \varpi - \left(\frac{2+l-\theta}{p-1} \right) \left[N + \theta - 2 - \left(\frac{2+l-\theta}{p-1} \right) \right] \varpi + \varpi^p = 0 \quad \text{on } S^{N-1}.$$

Theorem 1.2. *Assume that (1.3) holds, $p \neq \frac{N+2+2l-\theta}{N+\theta-2}$, and v is a positive solution of (1.1) in $\mathbb{R}^N \setminus B_R$ such that for some positive constant C ,*

$$|x|^{\frac{2+l-\theta}{p-1}} v(x) \leq C \quad \text{in } \mathbb{R}^N \setminus B_R.$$

Then either

$$|x|^{N-2+\theta} v(x) \rightarrow \gamma \quad \text{as } |x| \rightarrow \infty \text{ for some } \gamma > 0$$

or

$$r^{\frac{2+l-\theta}{p-1}} v(r\zeta) \rightarrow \varpi(\zeta) \quad \text{as } r \rightarrow \infty \text{ uniformly in } \zeta \in S^{N-1},$$

where $\varpi(\omega)$ is a positive solution of (1.4).

Remark 1.3. It is easily checked that the condition in Theorems A and B on ℓ , namely

$$(1.5) \quad \ell < \frac{2+\alpha}{p-1} \left(N - 2 - \frac{2+\alpha}{p-1} \right),$$

is equivalent to (1.3) with $\theta = -2\sigma$ and $l = \alpha - \sigma(p+1)$.

We now introduce the setting in which the finite Morse index theory for (1.1) will be developed. This is a crucial first step for the analysis of this paper. As mentioned before, we need to choose the setting with great care in order to make the notion of finite Morse index useful. In particular, we want the stability of a solution to (1.1) unchanged under various natural transformations, including (1.2), the Kelvin transformation (1.8) and the transformation (1.11) given below.

For $\theta \in \mathbb{R}^1$, we denote by $H^{1,\theta}(\Omega)$ the space of functions φ such that

$$|x|^{\frac{\theta}{2}}\varphi \in L^2(\Omega), \quad |x|^{\frac{\theta}{2}}|\nabla\varphi| \in L^2(\Omega),$$

with norm

$$\|\varphi\| = \left(\int_{\Omega} |x|^{\theta} (\varphi^2 + |\nabla\varphi|^2) dx \right)^{1/2}.$$

$H_{loc}^{1,\theta}(\Omega)$ is defined in the obvious way, and we use $H_c^{1,\theta}(\Omega)$ to denote the subspace of functions in $H^{1,\theta}(\Omega)$ which have compact supports in Ω . If $0 \notin \Omega$, clearly $H_{loc}^{1,\theta}(\Omega) = H_{loc}^1(\Omega)$ and $H_c^{1,\theta}(\Omega) = H_c^1(\Omega)$. If further $0 \notin \overline{\Omega}$ and Ω is bounded, then $H^{1,\theta}(\Omega) = H^1(\Omega)$.

Remark 1.4. (i) If $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ is bounded, then $u \in H^{1,\theta}(\Omega)$ if and only if $|x|^{\frac{\theta}{2}}u \in H^1(\Omega)$. This is a direct consequence of the identity

$$\nabla(|x|^{\frac{\theta}{2}}u) = |x|^{\frac{\theta}{2}}\nabla u + \frac{\theta}{2} \frac{x}{|x|^2} |x|^{\frac{\theta}{2}}u$$

and the fact that $|x|^{-1} \in L^2(\Omega)$ for $N \geq 3$ when Ω is bounded.

(ii) If $N = 2$ and $0 \in \Omega$, then $|x|^{-1} \notin L^2(\Omega)$, and the above statement is not true.

(iii) If $N \geq 2$, $u \in L_{loc}^{\infty}(\Omega)$ and $N + \theta > 2$, then $u \in H_{loc}^{1,\theta}(\Omega)$ if and only if $|x|^{\frac{\theta}{2}}u \in H_{loc}^1(\Omega)$. To see this, it suffices to check that $|x|^{\frac{\theta}{2}-1}u \in L_{loc}^2(\Omega)$ under the given conditions. Indeed, from $N + \theta > 2$ one obtains $|x|^{\frac{\theta}{2}-1} \in L_{loc}^2(\Omega)$, which implies $|x|^{\frac{\theta}{2}-1}u \in L_{loc}^2(\Omega)$ since by assumption $u \in L_{loc}^{\infty}(\Omega)$.

We say that v is a solution of (1.1) if $v \in H_{loc}^{1,\theta}(\Omega) \cap L_{loc}^{\infty}(\Omega)$ and

$$(1.6) \quad \int_{\Omega} \left(|x|^{\theta} \nabla v \cdot \nabla \phi - |x|^l |v|^{p-1} v \phi \right) = 0 \quad \forall \phi \in H_c^{1,\theta}(\Omega) \cap L_{loc}^{\infty}(\Omega).$$

Let us observe that if v is a solution of (1.1), then by standard elliptic regularity $v \in C^2(\Omega \setminus \{0\})$ and hence is a classical solution of (1.1) in $\Omega \setminus \{0\}$. In particular, $v \in C^2(\Omega)$ whenever $0 \notin \Omega$. If $0 \in \Omega$, then (1.6) has a hidden restriction on v at $x = 0$ since $\int_{\Omega} |x|^l |v|^{p-1} v \phi dx$ need not be defined for arbitrary $v \in H_{loc}^{1,\theta}(\Omega) \cap L_{loc}^{\infty}(\Omega)$ and $\phi \in H_c^{1,\theta}(\Omega) \cap L_{loc}^{\infty}(\Omega)$. However, this hidden restriction disappears when $N + \theta > 2$ and $l - \theta > -2$, since in such a case, $N + l > 0$ and $|x|^l \in L_{loc}^1(\Omega)$.

A solution v of (1.1) is said to be *stable* if

$$Q_v(\psi) := \int_{\Omega} \left(|x|^{\theta} |\nabla \psi|^2 - p |x|^l |v|^{p-1} \psi^2 \right) \geq 0 \quad \forall \psi \in H_c^{1,\theta}(\Omega) \cap L_{loc}^{\infty}(\Omega).$$

Similar to [4], we say a solution of (1.1) has *Morse index* $k \geq 0$ if k is the maximal dimension of all subspaces X_k of $H_c^{1,\theta}(\Omega) \cap L_{loc}^\infty(\Omega)$ such that $Q_v(\psi) < 0$ for any $\psi \in X_k \setminus \{0\}$. Thus v is stable if and only if it has Morse index 0. Moreover, if v has finite Morse index over the domain Ω , then there exists a compact subset \mathcal{K} of Ω such that v is stable over any domain $\Omega' \subset \Omega \setminus \mathcal{K}$.

The above setting allows us to establish the following integral estimate for stable solutions of (1.1), which is a key step for the success of this approach. This estimate is an extension of Proposition 4 in [14] (for $\theta = l = 0$) and Proposition 1.7 in [7] (for $\theta = 0$ and $l > -2$), albeit that we have added an extra term $|\psi| \frac{|\nabla \psi|}{|x|}$ in the right hand side. However, this extra term does not affect the key estimates in its applications, even for the special cases considered in [7] and [14].

Proposition 1.5. *Let Ω be a domain (bounded or not) of \mathbb{R}^N ($N \geq 2$). Let $v \in H_{loc}^{1,\theta}(\Omega) \cap L_{loc}^\infty(\Omega)$ be a stable solution of (1.1) with $p > 1$. Then for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1]$ and any integer $m \geq \max\{\frac{p+\gamma}{p-1}, 2\}$ there exists a constant $C > 0$ depending only on p, m, γ, l and θ such that*

$$(1.7) \quad \begin{aligned} & \int_{\Omega} \left(|x|^\theta |\nabla(|v|^{\frac{\gamma-1}{2}} v)|^2 + |x|^l |v|^{\gamma+p} \right) |\psi|^{2m} \\ & \leq C \int_{\Omega} |x|^{\frac{\theta(\gamma+p)-l(\gamma+1)}{p-1}} \left(|\nabla \psi|^2 + |\psi| |\Delta \psi| + |\psi| \frac{|\nabla \psi|}{|x|} \right)^{\frac{p+\gamma}{p-1}} \end{aligned}$$

for all test functions $\psi \in C_0^2(\Omega)$ satisfying $|\psi| \leq 1$ in Ω .

As in [7], the Kelvin transformation will be a useful tool in this paper. If v is a solution of (1.1) over $B_R \setminus \{0\}$ ($N \geq 2$), then the function w defined by the Kelvin transformation

$$(1.8) \quad w(y) = |x|^{N-2+\theta} v(x), \quad y = \frac{x}{|x|^2}$$

satisfies the equation

$$(1.9) \quad -\operatorname{div}(|y|^\theta \nabla w) = |y|^\beta |w|^{p-1} w \quad \text{for } y \in \mathbb{R}^N \setminus \overline{B_{1/R}},$$

with $\beta = (N-2+\theta)(p-1) - (4+l-2\theta)$. We notice that $\beta - \theta > -2$ when $p > \frac{N+l}{N-2+\theta}$.

We have the following proposition which shows that the Kelvin transformation in (1.8) keeps the stability of v .

Proposition 1.6. *A solution v of (1.1) is stable in $B_R \setminus \{0\}$ if and only if the function w obtained by the Kelvin transformation in (1.8) is a stable solution of (1.9) in $\mathbb{R}^N \setminus \overline{B_{1/R}}$.*

The next proposition discusses the stability property between solutions of (P) and (1.1). Recall that these two problems are related through $v(x) = |x|^\sigma u(x)$ with $\theta = -2\sigma$, $l = \alpha - \sigma(p+1)$ and $\sigma = \frac{1}{2}[N-2 - \sqrt{(N-2)^2 - 4\ell}]$.

We say u is a solution of (P) if $u \in H_{loc}^1(\Omega)$, $|x|^\sigma u \in L_{loc}^\infty(\Omega)$, and

$$\int_{\Omega} \left(\nabla u \cdot \nabla \phi - \ell |x|^{-2} u \phi - |x|^\alpha |u|^{p-1} u \phi \right) = 0 \quad \forall \phi \in H_c^1(\Omega) \text{ with } |x|^\sigma \phi \in L_{loc}^\infty(\Omega).$$

It is said to be *stable* if

$$\mathcal{Q}_u(\phi) := \int_{\Omega} \left(|\nabla \phi|^2 - \ell |x|^{-2} \phi^2 - p |x|^\alpha |u|^{p-1} \phi^2 \right) \geq 0$$

for all $\phi \in H_c^1(\Omega)$ with $|x|^\sigma \phi \in L_{loc}^\infty(\Omega)$.

We say that u has Morse index $k \geq 0$ if k is the maximal dimension of all subspaces Y_k of $Y := \{\phi \in H_c^1(\Omega) : |x|^\sigma \phi \in L_{loc}^\infty(\Omega)\}$ such that $\mathcal{Q}_u(\phi) < 0$ for any $\phi \in Y_k \setminus \{0\}$.

Proposition 1.7. *Let u be a solution of (P). Then $v(x) := |x|^\sigma u(x)$ is a stable solution of (1.1) if and only if u is a stable solution of (P).*

To introduce the other results of this paper, we need to define two critical powers for (1.1). In order to use calculations in [7] by similarity, in the following, we denote

$$N' = N + \theta \quad \text{and} \quad \tau = l - \theta$$

for fixed l and θ in \mathbb{R}^1 . We assume from now on that

$$(1.10) \quad N' > 2 \quad \text{and} \quad \tau > -2,$$

unless otherwise specified.

To better understand the above restriction on N' and τ , we make use of another transformation

$$(1.11) \quad z(y) = v(x), \quad y = \frac{x}{|x|^2}.$$

A simple calculation shows that under this transformation v is a solution to (1.1) if and only if z is a solution to

$$(1.12) \quad -\operatorname{div}(|y|^{\tilde{\theta}} \nabla z) = |y|^{\tilde{l}} |z|^{p-1} z, \quad |y|^{-2} y \in \Omega,$$

with

$$\tilde{\theta} = 4 - 2N - \theta, \quad \tilde{l} = -2N - l.$$

If we define

$$\tilde{N}' := N + \tilde{\theta}, \quad \tilde{\tau} := \tilde{l} - \tilde{\theta},$$

then $\tilde{N}' + N' = 4$ and $\tilde{\tau} + \tau = -4$. Thus

$$N' < 2 \quad \text{if and only if} \quad \tilde{N}' > 2,$$

and

$$\tau < -2 \quad \text{if and only if} \quad \tilde{\tau} > -2.$$

Moreover, the stability of the solution of (1.1) is unchanged under the transformation (1.11):

Proposition 1.8. *A solution v of (1.1) is stable in $B_R \setminus \{0\}$ if and only if the function z obtained by the transformation in (1.11) is a stable solution of (1.12) in $\mathbb{R}^N \setminus \overline{B_{1/R}}$.*

Thus for every result we obtain in the case of (1.10) there is a parallel result in the case of $N' < 2$ and $\tau < -2$ through the transformation (1.11).

We will show that if $N' \geq 2$ and $\tau \leq -2$, then problem (1.1) has no positive solution over any punctured ball $B_R \setminus \{0\}$ (see Theorem 4.2 below). This implies, by the Kelvin transformation, problem (1.1) has no positive solution over any exterior domain $\mathbb{R}^N \setminus B_R$ if $p \leq \frac{N'+\tau}{N'-2}$. This also implies, by the transformation (1.11), that problem (1.1) has no positive solution over any exterior domain $\mathbb{R}^N \setminus B_R$ if $N' \leq 2$ and $\tau \geq -2$.

For these reasons, the case $N' \geq 2$ and $\tau \leq -2$, and the case $N' \leq 2$ and $\tau \geq -2$, are not considered further.¹ Our focus will be mainly on the case (1.10).

Suppose (1.10) holds and let

$$f(p) = p \frac{2+\tau}{p-1} \left(N' - 2 - \frac{2+\tau}{p-1} \right).$$

Evidently,

$$f\left(\frac{N'+\tau}{N'-2}\right) = 0, \quad f(\infty) = (2+\tau)(N'-2).$$

Replacing (N, α) in the calculations on page 3285 of [7] by (N', τ) , we find that the equation $f(p) = \frac{(N'-2)^2}{4}$ always has a solution in the interval $(\frac{N'+\tau}{N'-2}, \frac{N'+2+2\tau}{N'-2})$. We denote this solution by $P_-(N', \tau)$. A simple calculation shows that $f(p) = \frac{(N'-2)^2}{4}$ is equivalent to

$$ap^2 - bp + cp = 0$$

with

$$(1.13) \quad \begin{cases} a = (N'-2)(N'-4\tau-10), \\ b = 2(N'-2)^2 - 4(\tau+2)(\tau+N'), \\ c = (N'-2)^2. \end{cases}$$

From this, we obtain

$$P_-(N', \tau) := \frac{(N'-2)^2 - 2(2+\tau)(N'+\tau) - 2(2+\tau)\sqrt{(2+\tau)(2N'+\tau-2)}}{(N'-2)(N'-4\tau-10)}$$

if $N' \neq 4\tau + 10$, and $P_-(N', \tau) = \frac{4}{3}$ if $N' = 4\tau + 10$. Moreover, when $2 < N' \leq 10 + 4\tau$, f has the property

$$(1.14) \quad \begin{cases} f(p) < \frac{(N'-2)^2}{4} & \text{for } 1 < p < P_-(N', \tau), \\ f(p) > \frac{(N'-2)^2}{4} & \text{for } p > P_-(N', \tau). \end{cases}$$

¹Note, however, for $N' \geq 2$ and $\tau \leq -2$, one may still consider (1.1) over an exterior domain, and for $N' \leq 2$ and $\tau \geq -2$, one may consider (1.1) over a punctured ball. But we will not pursue these cases here.

When $N' > 10 + 4\tau$, there exists a second root of $f(p) = \frac{(N'-2)^2}{4}$ in $(1, \infty)$, given by

$$P_+(N', \tau) := \frac{(N' - 2)^2 - 2(2 + \tau)(N' + \tau) + 2(2 + \tau)\sqrt{(2 + \tau)(2N' + \tau - 2)}}{(N' - 2)(N' - 4\tau - 10)},$$

and it has the properties

$$\frac{N' + 2 + 2\tau}{N' - 2} < P_+(N', \tau) < \infty,$$

and

$$(1.15) \quad \begin{cases} f(p) < \frac{(N'-2)^2}{4} & \text{for } p \in (1, P_-(N', \tau)) \cup (P_+(N', \tau), \infty), \\ f(p) > \frac{(N'-2)^2}{4} & \text{for } p \in (P_-(N', \tau), P_+(N', \tau)). \end{cases}$$

We will show that the number

$$p_c(N', \tau) = \begin{cases} \infty & \text{if } 2 < N' \leq 10 + 4\tau, \\ P_+(N', \tau) & \text{if } N' > 10 + 4\tau, \end{cases}$$

serves as a critical power for (1.1). The number

$$\tilde{p}_c(N', \tau) := P_-(N', \tau) < \frac{N' + 2 + 2\tau}{N' - 2} < p_c(N', \tau)$$

is also a critical value for (1.1).

The first important role played by $p_c(N', \tau)$ can be seen from the following theorem.

Theorem 1.9. *If $2 < N' \leq 10 + 4\tau$ and $p > 1$, or $N' > 10 + 4\tau$ and $1 < p < p_c(N', \tau)$, and if $v \in H_{loc}^{1,\theta}(\mathbb{R}^N) \cap L_{loc}^\infty(\mathbb{R}^N)$ is a stable solution of (1.1) (nonnegative or not) in \mathbb{R}^N ($N \geq 2$), then $v \equiv 0$; on the other hand, if $p \geq p_c(N', \tau)$, (1.1) admits a family of stable positive radial solutions in \mathbb{R}^N .*

For the special case $\theta = l = 0$, the above result was first obtained in [14]. When $\theta = 0$ and $l > -2$, it was proved in [7]. See [10] for the important role played by $p_c(N', \tau)$ on the behavior of radially symmetric solutions.

All the other results in this paper treat equations over a punctured domain or an exterior domain. We say that a positive solution v of (1.1) has an *isolated singularity* at 0 if Ω contains a punctured ball $B_r \setminus \{0\}$, $0 \notin \Omega$ and v tends to ∞ along some sequence $x_n \rightarrow 0$. If on the other hand $\lim_{|x| \rightarrow 0} v(x) = \gamma$ is a finite number and v becomes a positive solution of (1.1) over B_r upon defining $v(0) = \gamma$, we say that $x = 0$ is a removable singularity of v .

Let $\Omega^* \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded domain such that $0 \in \Omega^*$. A positive solution v of (1.1) in $\mathbb{R}^N \setminus \Omega^*$ is called a *fast decay* solution if $\lim_{|x| \rightarrow \infty} |x|^{N+\theta-2} v(x) = \gamma$ for some $\gamma > 0$.

The following two results give sufficient conditions to meet the requirements in Theorems 1.1 and 1.2. We note that in these two theorems, we have no restriction on θ and l .

Theorem 1.10. *Let $\Omega_0 \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain containing 0, and let v be a positive solution of (1.1) in $\Omega_0 \setminus \{0\}$ with arbitrary θ , $l \in \mathbb{R}^1$. If v has finite Morse index and $1 < p < p_c(N, 0)$, then there exist $C > 0$ and $\epsilon > 0$ such that*

$$|x|^{\frac{2+\tau}{p-1}} v(x) \leq C \quad \text{for } 0 < |x| < \epsilon.$$

Hence by Theorem 1.1, when $p \in (\frac{N'+\tau}{N'-2}, p_c(N, 0)) \setminus \{\frac{N'+2+2\tau}{N'-2}\}$,

$$(A_0) \quad \begin{cases} \text{either } v \text{ has a removable singularity at } x = 0, \text{ or} \\ r^{\frac{2+\tau}{p-1}} v(r\zeta) \rightarrow \varpi(\zeta) \text{ as } r \rightarrow 0 \text{ uniformly in } \zeta \in S^{N-1}, \end{cases}$$

where ϖ is a positive solution of (1.4).

Theorem 1.11. Let $\Omega_0 \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain containing 0, and let v be a positive solution of (1.1) in $\mathbb{R}^N \setminus \Omega_0$ with arbitrary θ , $l \in \mathbb{R}^1$. If v has finite Morse index and $1 < p < p_c(N, 0)$, then there exist $C > 0$ and $\epsilon > 0$ such that

$$|x|^{\frac{2+\tau}{p-1}} v(x) \leq C \quad \text{for } |x| > \epsilon^{-1}.$$

Hence by Theorem 1.2, when $p \in (\frac{N'+\tau}{N'-2}, p_c(N, 0)) \setminus \{\frac{N'+2+2\tau}{N'-2}\}$,

$$(A_\infty) \quad \begin{cases} \text{either } v \text{ is a fast decay solution, i.e., } \lim_{|x| \rightarrow \infty} |x|^{N'-2} v(x) = \gamma > 0, \text{ or} \\ r^{\frac{2+\tau}{p-1}} v(r\zeta) \rightarrow \varpi(\zeta) \text{ as } r \rightarrow \infty \text{ uniformly in } \zeta \in S^{N-1}, \end{cases}$$

where ϖ is a positive solution of (1.4).

For the special case $\theta = 0$ and $l > -2$, the above two theorems were first proved in [7].

With more restrictions on p , we can determine the alternatives in (A_0) and (A_∞) .

Theorem 1.12. Let $\Omega_0 \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain containing 0, and let v be a positive solution of (1.1) in $\Omega_0 \setminus \{0\}$. If v has finite Morse index and if

$$(1.16) \quad \tilde{p}_c(N', \tau) < p < \min\{p_c(N', \tau), p_c(N, 0)\}, \quad p \neq \frac{N' + 2 + 2\tau}{N' - 2},$$

then $x = 0$ must be a removable singularity of v .

On the other hand, for $p \geq p_c(N', \tau)$ or $p \in (\frac{N'+\tau}{N'-2}, \tilde{p}_c(N', \tau))$, (1.1) has a positive stable solution on $\mathbb{R}^N \setminus \{0\}$ with an isolated singularity at 0 (which is V_∞ given below).

Remark 1.13. We will show in Remark 4.1 that the function $p_c(\cdot, \tau)$ is a decreasing function for fixed τ and $p_c(N', \cdot)$ is an increasing function for fixed N' , as long as the value of the functions is finite (i.e., $N' > 10 + 4\tau$). Moreover, when $\tau = \frac{p-1}{2p+2\sqrt{p(p-1)}} \theta$, we have $p_c(N', \tau) = p_c(N, 0)$. Therefore

$$\min\{p_c(N', \tau), p_c(N, 0)\} = \begin{cases} p_c(N', \tau) & \text{if } \tau \leq \frac{p-1}{2p+2\sqrt{p(p-1)}} \theta, \\ p_c(N, 0) & \text{if } \tau > \frac{p-1}{2p+2\sqrt{p(p-1)}} \theta. \end{cases}$$

Note that the inequality $\tau \leq \frac{p-1}{2p+2\sqrt{p(p-1)}} \theta$ is equivalent to

$$(1.17) \quad l \leq \left(1 + \frac{p-1}{2p+2\sqrt{p(p-1)}}\right) \theta.$$

Theorem 1.14. Let $\Omega_0 \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain containing 0, and let v be a positive solution of (1.1) in $\mathbb{R}^N \setminus \Omega_0$. If v has finite Morse index and if (1.16) holds, then v must be a fast decay solution (i.e., $\lim_{|x| \rightarrow \infty} |x|^{N'-2} v(x) = \gamma > 0$).

On the other hand, for $p \geq p_c(N', \tau)$ or $p \in (\frac{N'+\tau}{N'-2}, \tilde{p}_c(N', \tau))$, (1.1) has a stable positive radial solution on $\mathbb{R}^N \setminus \{0\}$ which decays at the slower rate $|x|^{-\frac{2+\tau}{p-1}}$ at ∞ (which is V_∞ given below).

Remark 1.15. Theorems 1.12 and 1.14 indicate that the conclusions in Theorems 1.5 and 1.6 of [7] hold only for p in the range

$$\underline{p}(\alpha) < p < \bar{p}(\alpha^-), \quad p \neq \frac{N+2+2\alpha}{N-2}$$

instead of

$$\frac{N+\alpha}{N-2} < p < \bar{p}(\alpha^-), \quad p \neq \frac{N+2+2\alpha}{N-2}$$

as claimed there. The mistake in [7] is caused by the statement that

$$f\left(\frac{N+2+2\beta}{N-2}\right) > \frac{(N-2)^2}{4} \text{ implies } p > \underline{p}(\beta).$$

The above statement is true if β is independent of p , but $\beta = (N-2)(p-1) - (4+\alpha)$ in [7]. We also note that $\underline{p}(\alpha)$ is increasing in $\alpha \in (-2, \infty)$ instead of decreasing (as stated in [7]).

It is easily checked that

$$V_\infty(x) = C_0|x|^{-\frac{2+\tau}{p-1}}, \quad \text{with } C_0 = \left\{ \frac{2+\tau}{p-1} \left(N' - 2 - \frac{2+\tau}{p-1} \right) \right\}^{1/(p-1)},$$

is a positive radial solution of (1.1) over $\mathbb{R}^N \setminus \{0\}$ provided that $\tau > -2$ and $p > \frac{N'+\tau}{N'-2}$. For calculation convenience, we note that if $v = v(r)$ is a radial solution of (1.1), then $v(r)$ satisfies

$$v_{rr} + \frac{N'-1}{r}v_r + r^\tau|v|^{p-1}v = 0.$$

Moreover, we will show that V_∞ is the only positive radial solution of (1.1) over a punctured ball $B_R \setminus \{0\}$ that has a singularity at 0 if $p > \frac{N'+2+2\tau}{N'-2}$ (see Theorem 4.3 below).

Remark 1.16. Theorems 1.12 and 1.14 imply that, if $\tilde{p}_c(N', \tau) < p < p_c(N', \tau)$, $p \neq \frac{N'+2+2\tau}{N'-2}$ and $\tau \leq \frac{p-1}{2p+\sqrt{p(p-1)}}\theta$, then the Morse index of V_∞ is ∞ as a positive solution of (1.1) over any punctured ball $B_r \setminus \{0\}$, or over any $\mathbb{R}^N \setminus B_R$, but when $p \geq p_c(N', \tau)$ or $p \in (\frac{N'+\tau}{N'-2}, \tilde{p}_c(N', \tau)]$, the Morse index of V_∞ is reduced to 0. We do not know whether Theorems 1.12 and 1.14 still hold if $\min\{p_c(N', \tau), p_c(N, 0)\}$ in (1.16) is replaced by $p_c(N', \tau)$ when $\tau > \frac{p-1}{2p+\sqrt{p(p-1)}}\theta$.

The rest of the paper is organized in the following way. In Section 2, we give the proofs of Propositions 1.5, 1.6, 1.7 and 1.8. In Section 3, we prove Theorem 1.9. Section 4 is devoted to the proof of Theorems 1.10 and 1.11, while Theorems 1.12 and 1.14 are proved in Section 5, the last section of the paper.

2. PROOFS OF THE BASIC RESULTS

In this section, we collect the proofs of all the basic results which will serve as tools in the proofs of our other results.

2.1. Proof of Proposition 1.5. We follow the lines of the proof of Proposition 4 in [14] and Proposition 1.7 of [7], but with considerable modifications. We divide the proof into three steps.

Step 1. For any $\varphi \in C_0^2(\Omega)$,

$$(2.1) \quad \int_{\Omega} |x|^{\theta} |\nabla(|v|^{\frac{\gamma-1}{2}} v)|^2 \varphi^2 = \frac{(\gamma+1)^2}{4\gamma} \int_{\Omega} |x|^l |v|^{p+\gamma} \varphi^2 + \frac{\gamma+1}{4\gamma} \int_{\Omega} |v|^{\gamma+1} \operatorname{div}(|x|^{\theta} \nabla(\varphi^2)).$$

This is obtained by taking $\phi = |v|^{\gamma-1} v \varphi^2$ in (1.6).

Step 2. For any $\varphi \in C_0^2(\Omega)$, we have

$$(2.2) \quad \begin{aligned} & \left(p - \frac{(\gamma+1)^2}{4\gamma} \right) \int_{\Omega} |x|^l |v|^{\gamma+p} \varphi^2 \\ & \leq \int_{\Omega} |x|^{\theta} |v|^{\gamma+1} |\nabla \varphi|^2 + \frac{\gamma-1}{4\gamma} \int_{\Omega} |v|^{\gamma+1} \operatorname{div}(|x|^{\theta} \nabla(\varphi^2)). \end{aligned}$$

The function $\psi = |v|^{\frac{\gamma-1}{2}} v \varphi$ belongs to $H_c^{1,\theta}(\Omega) \cap L_{loc}^{\infty}(\Omega)$, thus it can be used as a test function in the quadratic inequality $Q_v(\psi) \geq 0$. Taking this test function and using (2.1), we can easily obtain (2.2).

Step 3. For any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1]$ and any $m \geq \max\{\frac{p+\gamma}{p-1}, 2\}$, there exists a constant $C > 0$ depending only on p, m, γ, l, θ such that

$$(2.3) \quad \int_{\Omega} |x|^l |v|^{p+\gamma} |\psi|^{2m} \leq C \int_{\Omega} |x|^{\frac{\theta(\gamma+p)-l(\gamma+1)}{p-1}} \left(|\nabla \psi|^2 + |\psi| |\Delta \psi| + |\psi| \frac{|\nabla \psi|}{|x|} \right)^{\frac{p+\gamma}{p-1}}$$

for all test function $\psi \in C_0^2(\Omega)$ satisfying $|\psi| \leq 1$ in Ω .

From (2.2) we see that for any $\varphi \in C_0^2(\Omega)$,

$$(2.4) \quad \eta \int_{\Omega} |x|^l |v|^{p+\gamma} \varphi^2 \leq \kappa \int_{\Omega} |v|^{\gamma+1} \operatorname{div}(|x|^{\theta} \nabla(\varphi^2)) + \int_{\Omega} |x|^{\theta} |v|^{\gamma+1} |\nabla \varphi|^2$$

with

$$\eta = p - \frac{(\gamma+1)^2}{4\gamma}, \quad \kappa = \frac{\gamma-1}{4\gamma}.$$

For any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1]$, an elementary analysis shows that $\eta > 0$.

For any $\psi \in C_0^2(\Omega)$ with $|\psi| \leq 1$ in Ω , we set $\varphi = \psi^m$. Since $m \geq 2$, the function φ belongs to $C_0^2(\Omega)$ and it follows from (2.2) that

$$I := \int_{\Omega} |x|^l |v|^{\gamma+p} |\psi|^{2m} \leq C_{m,p,\gamma,l,\theta}^1 \int_{\Omega} |x|^{\theta} |v|^{\gamma+1} |\psi|^{2m-2} \left(|\nabla \psi|^2 + |\psi| |\Delta \psi| + |\psi| \frac{|\nabla \psi|}{|x|} \right)$$

where $C_{m,p,\gamma,l,\theta}^1 > 0$ depends on m, p, γ, l, θ . An application of Hölder's inequality yields

$$\begin{aligned} & \int_{\Omega} |x|^{\theta} |v|^{\gamma+1} |\psi|^{2m-2} \left(|\nabla \psi|^2 + |\psi| |\Delta \psi| + |\psi| \frac{|\nabla \psi|}{|x|} \right) \\ & \leq I^{\frac{\gamma+1}{\gamma+p}} \left[\int_{\Omega} |x|^{\frac{\theta(\gamma+p)-l(\gamma+1)}{p-1}} |\psi|^{2(m-\frac{\gamma+p}{p-1})} \left(|\nabla \psi|^2 + |\psi| |\Delta \psi| + |\psi| \frac{|\nabla \psi|}{|x|} \right)^{\frac{p+\gamma}{p-1}} \right]^{\frac{p-1}{\gamma+p}} \end{aligned}$$

and hence

$$(2.5) \quad \int_{\Omega} |x|^{\theta} |v|^{\gamma+p} |\psi|^{2m} \leq C \int_{\Omega} |x|^{\frac{\theta(\gamma+p)-l(\gamma+1)}{p-1}} \left(|\nabla \psi|^2 + |\psi| |\Delta \psi| + |\psi| \frac{|\nabla \psi|}{|x|} \right)^{\frac{p+\gamma}{p-1}},$$

which proves (1.7) and Proposition 1.5. \square

2.2. Proof of Proposition 1.6. For any given $\psi \in H_c^1(B_R \setminus \{0\}) \cap L_{loc}^{\infty}(B_R \setminus \{0\})$, we define

$$\tilde{\psi}(y) = |x|^{N'-2} \psi(x), \quad y = \frac{x}{|x|^2}.$$

Clearly $\tilde{\psi} \in H_c^1(\mathbb{R}^N \setminus \overline{B_{1/R}}) \cap L_{loc}^{\infty}(\mathbb{R}^N \setminus \overline{B_{1/R}})$. (Recall that for this kind of domains $H_c^{1,\theta}(\Omega) = H_c^1(\Omega)$.) Moreover,

$$\begin{aligned} & \int_{\mathbb{R}^N \setminus \overline{B_{1/R}}} \left[|y|^{\theta} |\nabla_y \tilde{\psi}|^2 - p |y|^{\beta} |w|^{p-1} \tilde{\psi}^2 \right] dy \\ &= \int_{B_R \setminus \{0\}} \left[|x|^{-\theta} |\nabla_x (|x|^{N'-2} \psi)|^2 |x|^4 - p |x|^{2(N-2)} |x|^{4+l} |v|^{p-1} \psi^2 \right] |x|^{-2N} dx \\ &= \int_{B_R \setminus \{0\}} \left[|x|^{\theta} |\nabla_x \psi|^2 - p |x|^l |v|^{p-1} \psi^2 \right] dx \\ &\quad + \int_{B_R \setminus \{0\}} |x|^{4-\theta-2N} \left[(N'-2)^2 |x|^{2(N'-3)} \psi^2 + 2(N'-2) |x|^{2(N'-2)} \frac{x \cdot \nabla_x \psi}{|x|^2} \psi \right] dx \\ &= \int_{B_R \setminus \{0\}} \left[|x|^{\theta} |\nabla_x \psi|^2 - p |x|^l |v|^{p-1} \psi^2 \right] dx \\ &\quad + \int_{B_R \setminus \{0\}} |x|^{4-\theta-2N} \left[(N'-2)^2 |x|^{2(N'-3)} \psi^2 + (N'-2) |x|^{2(N'-3)} x \cdot \nabla_x (\psi^2) \right] dx. \end{aligned}$$

Using integration by parts we find

$$\begin{aligned} & \int_{B_R \setminus \{0\}} |x|^{4-\theta-2N} \left[(N'-2)^2 |x|^{2(N'-3)} \psi^2 + (N'-2) |x|^{2(N'-3)} x \cdot \nabla_x (\psi^2) \right] dx \\ &= (N'-2)^2 \int_{B_R \setminus \{0\}} |x|^{\theta-2} \psi^2 - (N'-2) \int_{B_R \setminus \{0\}} \operatorname{div} \left(|x|^{\theta} \frac{x}{|x|^2} \right) \psi^2 dx \\ &= 0. \end{aligned}$$

Hence

$$\int_{\mathbb{R}^N \setminus \overline{B_{1/R}}} \left[|y|^{\theta} |\nabla_y \tilde{\psi}|^2 - p |y|^{\beta} |w|^{p-1} \tilde{\psi}^2 \right] dy = \int_{B_R \setminus \{0\}} \left[|x|^{\theta} |\nabla_x \psi|^2 - p |x|^l |v|^{p-1} \psi^2 \right] dx.$$

The proposition clearly follows from this identity. \square

2.3. Proof of Proposition 1.7. Firstly we recall that $\theta = -2\sigma$, $l = \alpha - \sigma(p+1)$ and $\sigma = \frac{1}{2}[N-2 - \sqrt{(N-2)^2 - 4\ell}]$. Moreover $u = |x|^{-\sigma} v \in H_{loc}^1(\Omega)$, $v \in H_{loc}^{1,\theta}(\Omega) \cap L_{loc}^{\infty}(\Omega)$.

For $\phi \in H_c^1(\Omega)$ with $|x|^{\sigma} \phi \in L_{loc}^{\infty}(\Omega)$, we define $\tilde{\phi}(x) = |x|^{\sigma} \phi(x)$. Then by Remark 1.4, $\tilde{\phi} \in H_c^{1,\theta}(\Omega) \cap L_{loc}^{\infty}(\Omega)$. Moreover, with $l = \alpha - \sigma(p+1)$ and $\sigma^2 - (N-2)\sigma + \ell = 0$, we

have

$$\begin{aligned}
Q_v(\tilde{\phi}) &= \int_{\Omega} \left[|x|^{\theta} |\nabla \tilde{\phi}|^2 - p|x|^l |v|^{p-1} \tilde{\phi}^2 \right] \\
&= \int_{\Omega} \left[|x|^{-2\sigma} |\nabla \tilde{\phi}|^2 - (\sigma^2 - (N-2)\sigma + \ell) |x|^{-2\sigma-2} \tilde{\phi}^2 - p|x|^{\alpha} |u|^{p-1} \phi^2 \right] \\
&= \int_{\Omega} \left[|x|^{-2\sigma} |\nabla \tilde{\phi}|^2 + \sigma \tilde{\phi}^2 \nabla(|x|^{-2\sigma-2} x) + (\sigma^2 - \ell) |x|^{-2\sigma-2} \tilde{\phi}^2 - p|x|^{\alpha} |u|^{p-1} \phi^2 \right] \\
&= \int_{\Omega} \left[|x|^{-2\sigma} |\nabla \tilde{\phi}|^2 - 2\sigma |x|^{-2\sigma} \tilde{\phi} \nabla \tilde{\phi} \cdot \frac{x}{|x|^2} + (\sigma^2 - \ell) |x|^{-2\sigma-2} \tilde{\phi}^2 - p|x|^{\alpha} |u|^{p-1} \phi^2 \right] \\
&= \int_{\Omega} \left[|\nabla(|x|^{-\sigma} \tilde{\phi})|^2 - \ell |x|^{-2} |x|^{-2\sigma} \tilde{\phi}^2 - p|x|^{\alpha} |u|^{p-1} \phi^2 \right] \\
&= \int_{\Omega} \left[|\nabla \phi|^2 - \ell |x|^{-2} \phi^2 - p|x|^{\alpha} |u|^{p-1} \phi^2 \right] \\
&= Q_u(\phi).
\end{aligned}$$

The conclusion of the proposition follows easily from the above identity. \square

2.4. Proof of Proposition 1.8. This follows from a simple calculation. For any given $\psi \in H_c^1(B_R \setminus \{0\}) \cap L_{loc}^{\infty}(B_R \setminus \{0\})$, we define

$$\tilde{\psi}(y) = \psi(x), \quad y = \frac{x}{|x|^2}.$$

Clearly $\tilde{\psi} \in H_c^1(\mathbb{R}^N \setminus \overline{B_{1/R}}) \cap L_{loc}^{\infty}(\mathbb{R}^N \setminus \overline{B_{1/R}})$, and

$$\begin{aligned}
&\int_{\mathbb{R}^N \setminus \overline{B_{1/R}}} \left[|y|^{\tilde{\theta}} |\nabla_y \tilde{\psi}|^2 - p|y|^{\tilde{l}} |z|^{p-1} \tilde{\psi}^2 \right] dy \\
&= \int_{B_R \setminus \{0\}} \left[|x|^{-\tilde{\theta}} |\nabla_x \psi|^2 |x|^4 - p|x|^{-\tilde{l}} |v|^{p-1} \psi^2 \right] |x|^{-2N} dx \\
&= \int_{B_R \setminus \{0\}} \left[|x|^{\theta} |\nabla_x \psi|^2 - p|x|^l |v|^{p-1} \psi^2 \right] dx.
\end{aligned}$$

The conclusion of the proposition is a direct consequence of the above identity. \square

3. PROOF OF THEOREM 1.9

We need the following lemma.

Lemma 3.1. *Suppose that $p > \frac{N'+2+2\tau}{N'-2}$, $N' > 2$ and $\tau > -2$. Then for every $\kappa > 0$, problem (1.1) with $\Omega = \mathbb{R}^N$ has a unique positive radial solution v_{κ} satisfying $v(0) = \kappa$. Moreover, v_{κ} is of the form*

$$v_{\kappa}(r) = \kappa v_1(\kappa^{\frac{p-1}{\tau+2}} r)$$

where v_1 is the unique solution of the problem

$$(3.1) \quad \begin{cases} (r^{N-1+\theta} v'(r))' + r^{N-1+l} v^p(r) = 0, & r > 0, \\ v(0) = 1, & \lim_{r \rightarrow 0^+} r^{N-1+\theta} v'(r) = 0, \end{cases}$$

and v_{κ} has the properties:

(i) for every $\kappa > 0$,

$$(3.2) \quad \lim_{r \rightarrow \infty} r^{\frac{2+\tau}{p-1}} v_\kappa(r) = \left\{ \frac{2+\tau}{p-1} \left(N' - 2 - \frac{2+\tau}{p-1} \right) \right\}^{1/(p-1)}.$$

(ii) for $p \geq p_c(N', \tau)$,

$$(3.3) \quad v_\kappa(r) < V_\infty(r) := C_0 r^{-\frac{2+\tau}{p-1}} \quad \forall r > 0, \quad \forall \kappa > 0.$$

Proof. If $\theta = 0$, this is Lemma 4.1 in [7], which follows from results in [19, 20, 26]. Since the ODE satisfied by $u_\kappa(r)$ here is exactly the same as that satisfied by the radial solution in Lemma 4.1 of [7] once (N, α) there is replaced by (N', τ) , the conclusions here follow from the same reasoning as in [7] if we replace (N, α) there by (N', τ) .

The conclusions of this lemma are also contained in Corollary 1.3 of [10], where radial solutions of more general equations are considered. \square

Proof of Theorem 1.9. We first show the nonexistence of nontrivial stable solutions of (1.1) for $1 < p < p_c(N', \tau)$. Arguing indirectly we assume that $1 < p < p_c(N', \tau)$ and (1.1) has a solution $v \not\equiv 0$ that is stable. We are going to deduce a contradiction.

For every $R > 0$, we define the test function $\psi_R(x) = \varphi(\frac{|x|}{R})$, where $\varphi \in C^2(\mathbb{R})$, $0 \leq \varphi \leq 1$ everywhere on \mathbb{R} and

$$\varphi(t) = \begin{cases} 1 & \text{if } |t| \leq 1, \\ 0 & \text{if } |t| \geq 2. \end{cases}$$

We observe that for any $\gamma \in [1, 2p+2\sqrt{p(p-1)}-1)$ and any $m \geq \max\{\frac{p+\gamma}{p-1}, 2\}$, Proposition 1.5 gives

$$\begin{aligned} \int_{B_R} |x|^l |v|^{p+\gamma} &\leq C \int_{B_{2R} \setminus B_R} |x|^{\frac{\theta(\gamma+p)-l(\gamma+1)}{p-1}} \left[|\nabla \psi|^2 + |\psi| |\Delta \psi| + |\psi| \frac{|\nabla \psi|}{|x|} \right]^{\frac{\gamma+p}{p-1}} \\ &\leq C R^{N' - \frac{(2+\tau)\gamma+2p+\tau}{p-1}} \quad \forall R > 0. \end{aligned}$$

where C is a positive constant independent of R .

Consider the function

$$\Delta(N', p, \gamma, \tau) = N'(p-1) - (2+\tau)\gamma - 2p - \tau,$$

and define

$$\gamma(p) = 2p + 2\sqrt{p(p-1)} - 1, \quad \Gamma(p) = \frac{(2+\tau)\gamma(p) + 2p + \tau}{p-1}.$$

As in the proof of Theorem 2.1 in [7] we can rewrite $\Gamma(p)$ in the form

$$\Gamma(p) = 2(2+\tau) \left(1 + \frac{1}{p-1} + \sqrt{1 + \frac{1}{p-1}} \right) + 2$$

which shows that $\Gamma(p)$ is strictly decreasing in p for $p > 1$, with $\Gamma(1) = +\infty$ and $\Gamma(+\infty) = 10+4\tau$. Therefore $\Delta(N', p, \gamma(p), \tau) = (p-1)(N' - \Gamma(p)) < 0$ for all $p > 1$ when $N' \leq 10+4\tau$,

and for $N' > 10 + 4\tau$, there is a unique $p^* = p^*(\tau) \in (1, \infty)$ such that $N' = \Gamma(p^*)$ and

$$(p - p^*)\Delta(N', p, \gamma(p), \tau) > 0 \text{ for } p \in (1, \infty), p \neq p^*.$$

We note that $N' = \Gamma(p^*)$ is equivalent to

$$\left(\frac{N' - 2}{2 + \tau} - 2\right)p^* - \frac{N' - 2}{2 + \tau} = 2\sqrt{p^*(p^* - 1)}.$$

It follows that

$$p^* > \frac{N' - 2}{2 + \tau} \left(\frac{N' - 2}{2 + \tau} - 2\right)^{-1} > \frac{N' + 2 + 2\tau}{N' - 2} > \tilde{p}_c(N', \tau).$$

On the other hand, a simple calculation shows that the equation

$$\left[\left(\frac{N' - 2}{2 + \tau} - 2\right)p^* - \frac{N' - 2}{2 + \tau}\right]^2 = 4p^*(p^* - 1)$$

is equivalent to

$$a(p^*)^2 - bp^* + c = 0 \text{ with } a, b, c \text{ given by (1.13).}$$

Thus we necessarily have $p^* = p_c(N', \tau)$, and

$$\begin{aligned} \Delta(N', p, \gamma(p), \tau) &= 0 & \text{for } p = p_c(N', \tau); \\ \Delta(N', p, \gamma(p), \tau) &< 0 & \text{for } 1 < p < p_c(N', \tau). \end{aligned}$$

Since we have assumed $1 < p < p_c(N', \tau)$, we can choose $\gamma \in (1, \gamma(p))$ close enough to $\gamma(p)$ such that

$$N' - \frac{(2 + \tau)\gamma + 2p + \tau}{p - 1} < 0.$$

Fix such a γ and let $R \rightarrow +\infty$ in our earlier inequality, we conclude that

$$\int_{\mathbb{R}^N} |x|^l |v|^{\gamma+p} = 0.$$

This implies $|v|^{\gamma+p} \equiv 0$ in \mathbb{R}^N ; a contradiction.

Next we show that if $p \geq p_c(N', \tau)$ (which is possible only if $N' > 10 + 4\tau$), then for every $\kappa > 0$, the positive radial solution v_κ defined in Lemma 3.1 is a stable solution of (1.1).

We first show $v_\kappa \in H_{loc}^{1,\theta}(\mathbb{R}^N)$. We only need to show that for any $R > 1$,

$$\int_{B_R} |x|^\theta v_\kappa^2 < \infty, \quad \int_{B_R} |x|^\theta |\nabla v_\kappa|^2 < \infty.$$

Since $v_\kappa \in L_{loc}^\infty(\mathbb{R}^N)$, the first inequality is an easy consequence of the assumption that $N' = N + \theta > 2$. We now show that $\int_{B_R} |x|^\theta |\nabla v_\kappa|^2 dx < \infty$. It follows from the equation

of v_κ that $v'_\kappa(r) < 0$ for $r > 0$. Moreover,

$$\begin{aligned}
|v'_\kappa(r)| &= r^{1-N-\theta} \int_0^r s^{N-1+l} v_\kappa^p(s) ds \\
&\leq r^{1-N-\theta} \int_0^r s^{N-1+l} V_\infty^p(s) ds \quad \text{by (3.3)} \\
&= C_0^p r^{1-N-\theta} \int_0^r s^{N-1+l-\frac{p(2+\tau)}{p-1}} ds \\
&= C_{p,N',\tau} r^{1+\tau-\frac{p(2+\tau)}{p-1}} \quad (\text{note that } N-1+l-\frac{p(2+\tau)}{p-1} > -1 \text{ for } p > \frac{N'+\tau}{N'-2}).
\end{aligned}$$

Therefore, for any $R > 0$ and $p \geq p_c(N', \tau)$ ($> \frac{N'+2+2\tau}{N'-2}$), we have $N+1+\theta+2\tau-\frac{2p(2+\tau)}{p-1} > -1$ and

$$\int_{B_R} |x|^\theta |\nabla v_\kappa|^2 = \int_0^R r^{N-1+\theta} |v'_\kappa(r)|^2 dr \leq C_{p,N',\tau}^2 \int_0^R r^{N+1+\theta+2\tau-\frac{2p(2+\tau)}{p-1}} dr < \infty.$$

Since (3.3) holds, we have, for every $\psi \in C_0^1(\mathbb{R}^N)$,

$$\begin{aligned}
Q_{v_\kappa}(\psi) &= \int_{\mathbb{R}^N} |x|^\theta |\nabla \psi|^2 - p \int_{\mathbb{R}^N} |x|^l v_\kappa^{p-1} \psi^2 \\
&\geq \int_{\mathbb{R}^N} |x|^\theta |\nabla \psi|^2 - p \int_{\mathbb{R}^N} |x|^l V_\infty^{p-1} \psi^2 \\
&= \int_{\mathbb{R}^N} |x|^\theta |\nabla \psi|^2 - \int_{\mathbb{R}^N} p C_0^{p-1} |x|^l |x|^{-(2+\tau)} \psi^2 \\
&= \int_{\mathbb{R}^N} |x|^\theta |\nabla \psi|^2 - \int_{\mathbb{R}^N} p C_0^{p-1} |x|^{-(2-\theta)} \psi^2.
\end{aligned}$$

By the Caffarelli-Kohn-Nirenberg inequality [2],

$$\left(\int_{\mathbb{R}^N} \frac{|\psi|^q}{|x|^{bq}} dx \right)^{2/q} \leq C(N, a, b) \int_{\mathbb{R}^N} \frac{|\nabla \psi|^2}{|x|^{2a}} dx,$$

where $C(N, a, b)$ is a positive constant and

$$-\infty < a < \frac{N-2}{2}, \quad a \leq b \leq a+1, \quad q = \frac{2N}{N-2+2(b-a)}.$$

In our case here,

$$a = -\frac{\theta}{2}, \quad b = 1 - \frac{\theta}{2} = 1 + a, \quad q = 2,$$

and by [3], $C(N, -\frac{\theta}{2}, 1 - \frac{\theta}{2})$ has the optimal value $\frac{4}{(N'-2)^2}$. Therefore

$$\int_{\mathbb{R}^N} \frac{|\psi|^2}{|x|^{2-\theta}} dx \leq \frac{4}{(N'-2)^2} \int_{\mathbb{R}^N} |x|^\theta |\nabla \psi|^2 dx,$$

and

$$(3.4) \quad \int_{\mathbb{R}^N} |x|^\theta |\nabla \psi|^2 - \int_{\mathbb{R}^N} p C_0^{p-1} |x|^{-(2-\theta)} \psi^2 \geq \left(\frac{(N'-2)^2}{4} - p C_0^{p-1} \right) \int_{\mathbb{R}^N} |x|^{-(2-\theta)} \psi^2 \geq 0,$$

since

$$\frac{(N'-2)^2}{4} - p C_0^{p-1} = \frac{(N'-2)^2}{4} - f(p) \geq 0 \quad \text{for } p \geq p_c(N', \tau).$$

Thus $Q_{v_\kappa}(\psi) \geq 0$. This means that v_κ is a stable solution of (1.1). This completes the proof. \square

4. ASYMPTOTIC BOUNDS AND RELATED RESULTS

In this section, we supply the proofs of Theorems 1.10 and 1.11, and also prove the necessity of the assumption $\tau > -2$ and the uniqueness of the radial solution V_∞ .

4.1. Proof of Theorem 1.10. Since v has finite Morse index, it is stable outside a compact subset of Ω and hence there exists $R_* > 0$ small such that v is stable in $B_{R_*} \setminus \{0\}$.

Step 1. Suppose that v is a stable positive solution of (1.1) in $B_{R_*} \setminus \{0\}$. Then for every $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and every open ball $B_R(y)$ with $0 < |y| < \frac{4}{5}R_*$ and $R = \frac{|y|}{4}$, we have

$$(4.1) \quad \int_{B_R(y)} |x|^l v^{\gamma+p} \leq CR^{N' - \frac{(2+\tau)\gamma+2p+\tau}{p-1}},$$

where C is a positive constant depending on m, p, N', τ but not on y .

Since v is stable in $B_{R_*} \setminus \{0\}$, Proposition 1.5 holds when $\Omega = B_{R_*} \setminus \{0\}$. We fix a function $\varphi_0 \in C^2(\mathbb{R})$ satisfying $0 \leq \varphi_0 \leq 1$ everywhere on \mathbb{R} and

$$\varphi_0(t) = \begin{cases} 0 & \text{if } |t| \leq 1, \\ 1 & \text{if } |t| \geq 2. \end{cases}$$

We then apply Proposition 1.5 with $m = 1 + \max\{\frac{p+\gamma}{p-1}, 2\}$ and test function $\psi(x) := \varphi_0(\frac{|x-y|}{R})$ and obtain (4.1) as we did in the proof of Theorem 1.9.

Step 2. Suppose that v is a stable solution of (1.1) in $B_{R_*} \setminus \{0\}$. Then if $1 < p < p_c(N, 0)$, there exists a small $\epsilon_0 = \epsilon_0(p) > 0$ such that for every $\epsilon \in [0, \epsilon_0]$ and every open ball $B_{2R}(y)$ with $0 < |y| \leq \frac{2}{3}R_*$ and $R = |y|/8$, we have

$$(4.2) \quad R^{-\frac{N\theta}{2-\epsilon}} \int_{B_{2R}(y)} \left(|x|^l v^{p-1}\right)^{\frac{N}{2-\epsilon}} \leq CR^{N - \frac{2N}{2-\epsilon}},$$

where C is a positive constant depending on m, p, N, τ but not on y and ϵ .

Let us recall that for

$$\Delta(N', p, \gamma, \tau) = N'(p-1) - (2+\tau)\gamma - 2p - \tau$$

and

$$\gamma(p) = 2p + 2\sqrt{p(p-1)} - 1,$$

we have

$$\begin{aligned} \Delta(N', p, \gamma(p), \tau) &= 0 & \text{for } p = p_c(N', \tau); \\ \Delta(N', p, \gamma(p), \tau) &< 0 & \text{for } 1 < p < p_c(N', \tau). \end{aligned}$$

Taking $\tau = 0$ we obtain

$$(4.3) \quad \Delta(N, p, \gamma(p), 0) = N(p-1) - 2(\gamma(p) + p) < 0 \quad \text{for } 1 < p < p_c(N, 0).$$

Thus we can fix $\gamma_* = \gamma_*(p) \in (1, \gamma(p))$ such that

$$(4.4) \quad \frac{p + \gamma_*}{(p-1)N/2} > 1.$$

It is seen from (4.4) that we can find $\epsilon_0 = \epsilon_0(p) > 0$ sufficiently small so that

$$\frac{p + \gamma_*}{(p-1)\rho} > 1 \quad \forall \rho \in \left[\frac{N}{2}, \frac{N}{2 - \epsilon_0}\right].$$

Fix such a ρ and set

$$\xi = \frac{p + \gamma_*}{(p-1)\rho}.$$

By Hölder's inequality and (4.1),

$$\begin{aligned} \int_{B_{2R}(y)} (|x|^l v^{p-1})^\rho &\leq \left(\int_{B_{2R}(y)} |x|^l v^{\gamma_* + p} \right)^{1/\xi} \left(\int_{B_{2R}(y)} |x|^{\frac{l(\rho\xi-1)}{\xi-1}} \right)^{(\xi-1)/\xi} \\ &\leq CR^{(N' - \frac{(2+\tau)\gamma_* + 2p + \tau}{p-1})\frac{1}{\xi}} R^{(N + \frac{l(\rho\xi-1)}{\xi-1})\frac{\xi-1}{\xi}} \\ &= CR^{N-2\rho + \frac{(p+\gamma_*)\theta}{(p-1)\xi}}, \end{aligned}$$

which implies that

$$(4.5) \quad R^{-\theta\rho} \int_{B_{2R}(y)} (|x|^l v^{p-1})^\rho \leq CR^{N-2\rho},$$

and (4.2) follows if we take $\rho = \frac{N}{2-\epsilon}$.

Step 3. Harnack inequality: Under the conditions of Step 2, there exists a positive constant K such that

$$(4.6) \quad \max_{|x|=r} v(x) \leq K \min_{|x|=r} v(x) \quad \forall r \in (0, R_*].$$

Regarding $v = v(x)$ as a solution of the equation

$$\operatorname{div}(|x|^\theta \nabla v) + d(x)v = 0$$

with $d(x) = |x|^l v^{p-1}(x)$, in view of (4.2), we can apply Harnack's inequality on each ball $B_R(y)$ with $0 < |y| < \frac{2}{3}R_*$, $R = \frac{|y|}{8}$, to obtain

$$(4.7) \quad \sup_{B_R(y)} v \leq K \inf_{B_R(y)} v,$$

where K depends on N', m, p, τ and $R^\epsilon \|R^{-\theta} d\|_{L^{\frac{N}{2-\epsilon}}(B_{2R}(y))}$ (see [17] p. 209). (Note that for $x \in B_{2R}(y)$, $|y| - |x - y| \leq |x| \leq |y| + |x - y|$ and thus $6R \leq |x| \leq 10R$. This implies that $|x|^\theta \geq 6^\theta R^\theta$ provided $\theta \geq 0$; $|x|^\theta \geq 10^\theta R^\theta$ provided $\theta < 0$. Therefore, the λ in [17] is $6^\theta R^\theta$ or $10^\theta R^\theta$.) Due to (4.2),

$$R^\epsilon \|R^{-\theta} d\|_{L^{\frac{N}{2-\epsilon}}(B_{2R}(y))} \leq R^\epsilon CR^{-\epsilon} = C.$$

Therefore, K is independent of R . Given any $r \in (0, \frac{2}{3}R_*]$, the sphere $\{|x| = r\}$ can be covered by a finite number of balls of the form $B_R(y)$ with $|y| = r$ and $R = |y|/8 = r/8$, and this finite number is independent of r . Therefore, by enlarging K in (4.7) properly, we have

$$\max_{|x|=r} v(x) \leq K \min_{|x|=r} v(x) \quad \forall r \in \left(0, \frac{2}{3}R_*\right].$$

Since v is positive and continuous in $\{\frac{2}{3}R_* \leq |x| \leq R_*\}$, by further enlarging K if necessary, we can guarantee that the above inequality holds for all $r \in (0, R_*]$, and (4.6) is proved.

Step 4. Under the conditions of Step 2, there exists a positive constant C such that

$$(4.8) \quad v(x) \leq C|x|^{-\frac{2+\tau}{p-1}} \quad \forall x \in B_{R_*} \setminus \{0\}.$$

From (4.2) with $\epsilon = 0$ we obtain, for $0 < |y| < \frac{2}{3}R_*$ and $R = |y|/8$,

$$\vartheta \left[\inf_{B_R(y)} v \right]^{\frac{N(p-1)}{2}} R^{N(1+\frac{\tau}{2})} \leq \int_{B_R(y)} \left[R^{-\theta} |x|^l v^{p-1} \right]^{\frac{N}{2}} \leq C,$$

where $\vartheta := \vartheta(N)$ is a positive constant independent of y and v . It follows that

$$\inf_{B_R(y)} v \leq \left(\frac{C}{\vartheta} \right)^{\frac{2}{N(p-1)}} R^{-\frac{2+\tau}{p-1}}.$$

We can now apply (4.6) to obtain

$$\sup_{B_R(y)} v \leq K \left(\frac{C}{\vartheta} \right)^{\frac{2}{N(p-1)}} R^{-\frac{2+\tau}{p-1}}.$$

In particular,

$$v(y) \leq K \left(\frac{C}{\vartheta} \right)^{\frac{2}{N(p-1)}} R^{-\frac{2+\tau}{p-1}} = C_1 |y|^{-\frac{2+\tau}{p-1}}$$

for all y satisfying $0 < |y| \leq \frac{2}{3}R_*$. Since both $v(y)$ and $|y|^{-\frac{2+\tau}{p-1}}$ are positive and continuous on $\{\frac{2}{3}R_* \leq |y| \leq R_*\}$, by enlarging C_1 if necessary, we have

$$(4.9) \quad v(y) \leq C_1 |y|^{-\frac{2+\tau}{p-1}} \quad \text{for all } y \text{ satisfying } 0 < |y| \leq R_*.$$

The proof is complete. \square

Remark 4.1. The condition $1 < p < p_c(N, 0)$ in Theorem 1.10 is only used to obtain (4.4). We may attempt to replace it by other conditions. For example, since $\Delta(N', p, \gamma(p), \tau) < 0$ for $1 < p < p_c(N', \tau)$, we see that

$$(4.10) \quad N(p-1) - 2(\gamma(p) + p) < (\gamma(p) + 1) \left[\tau - \frac{(p-1)}{(1+\gamma(p))} \theta \right] \leq 0 \quad \text{for } 1 < p < p_c(N', \tau)$$

provided $\tau - \frac{(p-1)}{(1+\gamma(p))} \theta \leq 0$. Therefore, we can fix $\gamma_* \in [1, \gamma(p))$ such that (4.4) holds provided

$$(A) \quad \tau \leq \frac{(p-1)}{2p + 2\sqrt{p(p-1)}} \theta \quad \text{and} \quad 1 < p < p_c(N', \tau).$$

However, it is easy to see that condition (A) is more restrictive than requiring $1 < p < p_c(N', 0)$, because we will show below that the function $p_c(N', \cdot)$ is increasing for any fixed N' , and thus $\tau \leq \frac{(p-1)}{2p+2\sqrt{p(p-1)}}\theta$ implies

$$(4.11) \quad p_c(N', \tau) \leq p_c\left(N', \frac{(p-1)}{2p+2\sqrt{p(p-1)}}\theta\right) = p_c(N, 0).$$

To see the equality above, we note that if $\tilde{\tau} = \frac{(p-1)}{2p+2\sqrt{p(p-1)}}\theta$, then

$$\begin{aligned} \Delta(N', p, \gamma(p), \tilde{\tau}) &= N(p-1) - 2(p + \gamma(p)) - (1 + \gamma(p))\left(\tilde{\tau} - \frac{(p-1)}{1 + \gamma(p)}\theta\right) \\ &= N(p-1) - 2(p + \gamma(p)) \\ &= \Delta(N', p, \gamma(p), 0). \end{aligned}$$

Hence from $\Delta(N', p_c(N', \tilde{\tau}), \gamma(p_c(N', \tilde{\tau})), \tilde{\tau}) = 0$ we obtain

$$N[p_c(N', \tilde{\tau}) - 1] - 2[p_c(N', \tilde{\tau}) + \gamma(p_c(N', \tilde{\tau}))] = 0$$

and thus $p_c(N', \tilde{\tau}) = p_c(N, 0)$.

On the other hand, if

$$(B) \quad \tau > \tilde{\tau} = \frac{(p-1)}{2p+2\sqrt{p(p-1)}}\theta \quad \text{and} \quad 1 < p < p_c(N', \tau),$$

then

$$p_c(N', \tau) > p_c(N', \tilde{\tau}) = p_c(N, 0).$$

We now show that $p_c(N', \tau)$ is decreasing in N' and increasing in τ . Recall that, for $N' > 4\tau + 10$, $p_c(N', \tau) \in (1, \infty)$ is the unique solution of

$$N' = \Gamma(p) = 2(2 + \tau) \left(1 + \frac{1}{p-1} + \sqrt{1 + \frac{1}{p-1}}\right) + 2,$$

which is equivalent to

$$\frac{N' - 2}{2 + \tau} = 2 \left(1 + \frac{1}{p-1} + \sqrt{1 + \frac{1}{p-1}}\right).$$

Since the term on the left hand side is increasing in N' and decreasing in τ , while the term on the right hand side is a decreasing function of p , it follows immediately that $p_c(N', \tau)$ is increasing in τ and decreasing in N' .

4.2. Proof of Theorem 1.11. Since v has finite Morse index, it is stable outside a compact subset of Ω and hence there exists $R_* > 0$ large such that v is stable in $\mathbb{R}^N \setminus \overline{B_{R_*}}$.

Define

$$w(y) = |x|^{N'-2}v(x), \quad y = \frac{x}{|x|^2}.$$

Then w satisfies

$$(4.12) \quad -\operatorname{div}(|y|^\theta \nabla w) = |y|^\beta w^p \quad \text{in } B_{1/R_*} \setminus \{0\},$$

with

$$\tau' := \beta - \theta = (N' - 2)(p - 1) - (4 + \tau) > -2 \text{ if } p > (N' + \tau)/(N' - 2).$$

By Proposition 1.6, w is a stable positive solution of (4.12). Therefore when $p \in \left(\frac{N'+\tau}{N'-2}, p_c(N', 0)\right)$, we can apply Theorem 1.10 to (4.12) to conclude that

$$|y|^{\frac{2+\beta-\theta}{p-1}} w(y) \leq C \quad \text{for all small } |y| > 0,$$

which is equivalent to

$$|x|^{\frac{2+\tau}{p-1}} v(x) \leq C \quad \text{for all large } |x| > 0.$$

It remains to consider the case that $p \in \left(1, \frac{N'+\tau}{N'-2}\right]$, which implies that $\tau' \leq -2$. By Theorem 4.2 below, in this case, (4.12) does not have a positive solution over any punctured ball $B_R \setminus \{0\}$, which implies that (1.1) has no positive solution over any exterior domain. Therefore there is nothing to prove for this case. \square

4.3. Related results. The next result reveals the role played by the condition $\tau > -2$.

Theorem 4.2. *For $N' \geq 2$ and $\tau \leq -2$, problem (1.1) does not admit a positive solution over any punctured ball $B_R \setminus \{0\} \subset \mathbb{R}^N$ ($N \geq 2$).*

Proof. We argue indirectly by assuming that $u \in C^2(B_R \setminus \{0\})$ is a positive solution of (1.1). Using spherical coordinates to write $v(x) = v(r, \omega)$ with $r = |x|$ and $\omega = \frac{x}{|x|}$, we have

$$v_{rr} + \frac{N' - 1}{r} v_r + \frac{1}{r^2} \Delta_{S^{N-1}} v = -r^\tau v^p.$$

This equation is exactly the same as that in Theorem 2.3 of [7] when (N, α) there is replaced by (N', τ) here. Since $N' \geq 2$, the arguments in the proof of Theorem 2.3 in [7] lead to a contradiction. The proof is thus complete. \square

Similarly, a positive radial solution $v(r)$ of (1.1) satisfies

$$v_{rr} + \frac{N' - 1}{r} v_r = -r^\tau v^p,$$

which is exactly the same as that satisfied by $u(r)$ in Theorem 2.4 of [7] with (N, α) there being replaced by (N', τ) here. Thus we have the following analogue of Theorem 2.4 of [7].

Theorem 4.3. *Let $v = v(r)$ be a positive radial solution of (1.1) over $B_R \setminus \{0\}$ with $\lim_{r \rightarrow 0} v(r) = \infty$ and $p > \frac{N'+2+2\tau}{N'-2}$. Then*

$$v(r) \equiv V_\infty(r).$$

This theorem implies that $V_\infty(r)$ is the unique positive radial singular solution of (1.1) over any B_R when $p > \frac{N'+2+2\tau}{N'-2}$.

5. EXACT ASYMPTOTIC BEHAVIOR

In this section, we prove Theorems 1.12 and 1.14. We first prove the results for $p > \frac{N'+2+2\tau}{N'-2}$. Then we make use of the Kelvin transformation to cover the full range of p .

Theorem 5.1. *Let Ω_0 be a bounded domain in \mathbb{R}^N ($N \geq 2$) containing 0, and let v be a positive solution of (1.1) with $\Omega = \Omega_0 \setminus \{0\}$. If v has finite Morse index, then $x = 0$ must be a removable singularity of v provided that*

$$(5.13) \quad \frac{N' + 2 + 2\tau}{N' - 2} < p < \min\{p_c(N', \tau), p_c(N, 0)\}.$$

On the other hand, if $p \geq p_c(N', \tau)$, then problem (1.1) has a stable positive solution with an isolated singularity at 0.

Proof. A direct calculation shows that, as long as $N' > 2$ and $p > \frac{N'+\tau}{N'-2}$,

$$(5.14) \quad V_\infty(x) := C_0 |x|^{-\frac{2+\tau}{p-1}}, \quad C_0 = \left\{ \frac{2+\tau}{p-1} \left(N' - 2 - \frac{2+\tau}{p-1} \right) \right\}^{1/(p-1)}$$

is a positive solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$, with 0 an isolated singularity.

Moreover, when $p \geq p_c(N', \tau)$, it follows from (3.4) that for every $\psi \in C_0^1(\mathbb{R}^N)$,

$$Q_{V_\infty}(\psi) = \int_{\mathbb{R}^N} [|x|^\theta |\nabla \psi|^2 - p |x|^l V_\infty^{p-1} \psi^2] \geq 0,$$

that is, V_∞ is a stable solution of (1.1) on $\mathbb{R}^N \setminus \{0\}$. In particular, it is a stable positive solution of (1.1) in Ω .

Next we suppose that (5.13) holds and that v is a positive solution of (1.1) with finite Morse index. For p in this range, Theorem 1.10 applies and hence there exist $C > 0$ and small $r_0 > 0$ such that

$$(5.15) \quad |x|^{\frac{2+\tau}{p-1}} v(x) \leq C \quad \text{for } 0 < |x| < r_0.$$

Hence we can apply Theorem 1.1 to conclude that v either has a removable singularity at $x = 0$ or

$$(5.16) \quad C_1 \leq |x|^{\frac{2+\tau}{p-1}} v(x) \leq C_2$$

for some $C_1, C_2 > 0$ and small positive $|x|$, say $0 < |x| < R_0$. Thus, to complete the proof, it suffices to show that (5.16) does not hold.

Arguing indirectly, we suppose that (5.16) holds, and then derive a contradiction. Since v has finite Morse index, we may assume that v is stable in $B_{R_*} \setminus \{0\}$ for some sufficiently small $R_* > 0$. We divide our arguments below into two steps.

Step 1. Suppose that $N' > 2$, $\tau > -2$, $p > 1$ and v is a stable positive solution of (1.1) in $B_{R_*} \setminus \{0\}$. Then there exists $R_0 \in (0, R_*)$ such that for every $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$

and every $r \in (0, R_0/2)$, we have

$$(5.17) \quad \int_{r < |x| < R_0} |x|^l v^{\gamma+p} \leq C + Dr^{N' - \frac{(2+\tau)\gamma+2p+\tau}{p-1}}$$

where C and D are positive constants depending on m, p, N', τ, R_0, R_* but not on r .

Since v is stable in $B_{R_*} \setminus \{0\}$, Proposition 1.5 holds with $\Omega = B_{R_*} \setminus \{0\}$. To choose a suitable test function for our purpose here, we fix a function $\varphi_0 \in C^2(\mathbb{R})$ as in the proof of Theorem 1.10 and choose another function ϱ_0 such that $\varrho_0 \in C^2(\mathbb{R})$, $0 \leq \varrho_0 \leq 1$ everywhere on \mathbb{R} and

$$\varrho_0(t) = \begin{cases} 1 & \text{if } t \leq R_0, \\ 0 & \text{if } t \geq (R_0 + R_*)/2. \end{cases}$$

For every $r \in (0, R_0/2)$, we define ξ_r as follows

$$\xi_r(x) = \begin{cases} \varrho_0(|x|) & \text{if } |x| \geq R_0/2, \\ \varphi_0(\frac{2|x|}{r}) & \text{if } |x| \leq R_0/2. \end{cases}$$

Clearly ξ_r belongs to $C_0^2(B_{R_*} \setminus \{0\})$ and satisfies $0 \leq \xi_r \leq 1$ everywhere on \mathbb{R}^N . We now choose $m = 1 + \max\{\frac{p+\gamma}{p-1}, 2\}$ and apply Proposition 1.5 with $\Omega = B_{R_*} \setminus \{0\}$ and $\psi = \xi_r$ to obtain

$$\begin{aligned} & \int_{r/2 < |x| < R_0} |x|^l v^{\gamma+p} \\ & \leq C \int_{\mathbb{R}^N} |x|^{\frac{\theta(p+\gamma)-l(\gamma+1)}{p-1}} \left(|\nabla \xi_r|^2 + |\xi_r| |\Delta \xi_r| + |\xi_r| \frac{|\nabla \xi_r|}{|x|} \right)^{\frac{p+\gamma}{p-1}} \\ & \leq \hat{C} \left[\int_{R_0 \leq |x| \leq R_*} |x|^{\frac{\theta(p+\gamma)-l(\gamma+1)}{p-1}} \left(|\varrho_0'(|x|)|^2 + \varrho_0(|x|) |\varrho_0''(|x|)| + \frac{|\varrho_0'(|x|)|}{|x|} \right)^{\frac{p+\gamma}{p-1}} \right. \\ & \quad \left. + \int_{\frac{r}{2} \leq |x| \leq r} |x|^{\frac{\theta(p+\gamma)-l(\gamma+1)}{p-1}} \left(r^{-2} |\varphi_0'(2|x|/r)|^2 \right. \right. \\ & \quad \left. \left. + r^{-2} \varphi_0(2|x|/r) |\varphi_0''(2|x|/r)| + 2r^{-2} |\varphi_0'(2|x|/r)| \right)^{\frac{p+\gamma}{p-1}} \right] \\ & \leq C_1 + C_2 r^{N' - \frac{(2+\tau)\gamma+2p+\tau}{p-1}} \end{aligned}$$

for all $r \in (0, R_0/2)$ and all $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$. Hence the desired integral estimate (5.17) holds.

Step 2. Reaching a contradiction when (5.13) holds.

Recall that

$$(5.18) \quad \Delta(N', p, \gamma(p), \tau) < 0 \quad \text{for } 1 < p < p_c(N', \tau).$$

Hence

$$(5.19) \quad \Delta(N', p, \gamma(p), \tau) < 0 \quad \text{when (5.13) holds.}$$

On the other hand,

$$(5.20) \quad \Delta(N', p, 1, \tau) = (N' - 2)p - (N' + 2) - 2\tau \geq 0 \quad \text{if } p \geq \frac{N' + 2 + 2\tau}{N' - 2}.$$

Therefore, under our assumption on p , we can find $\gamma_0 \in [1, \gamma(p))$ such that $\Delta(N', p, \gamma_0, \tau) = 0$, that is,

$$N' - \frac{(2 + \tau)\gamma_0 + 2p + \tau}{p - 1} = 0.$$

Choosing $\gamma = \gamma_0$ in (5.17), we obtain

$$\int_{\{r < |x| < R_0\}} |x|^l v^{\gamma_0 + p} \leq C + D.$$

On the other hand, using (5.16) we deduce

$$\begin{aligned} \int_{\{r < |x| < R_0\}} |x|^l v^{\gamma_0 + p} &\geq C_1^{\gamma_0 + p} \int_{\{r < |x| < R_0\}} |x|^{l - \frac{2 + \tau}{p - 1}(\gamma_0 + p)} = C_1^{p + \gamma_0} \int_r^{R_0} s^{-1} ds \\ &= C_1^{\gamma_0 + p} \log(R_0/r) \rightarrow \infty \quad \text{as } r \rightarrow 0^+, \end{aligned}$$

a contradiction. This completes the proof. \square

Theorem 5.2. *Suppose that Ω_0 is a bounded domain containing 0 and the condition (5.13) in Theorem 5.1 holds. If v is a positive solution of (1.1) in $\Omega := \mathbb{R}^N \setminus \Omega_0$ that has finite Morse index, then it must be a fast decay solution.*

On the other hand, if $p \geq p_c(N', \tau)$, then (1.1) admits a stable positive solution decaying at the slower rate $|x|^{-\frac{2 + \tau}{p - 1}}$ at infinity.

Proof. If $p \geq p_c(N', \tau)$, we already know from the proof of Theorem 5.1 that V_∞ is a stable positive solution of (1.1) over Ω with slow decay at infinity.

Next we suppose that (5.13) holds and v is a positive solution of (1.1) with finite Morse index. Therefore, Theorem 1.11 applies and there exists $C > 0$ and large $R_* > 0$ such that

$$(5.21) \quad |x|^{\frac{2 + \tau}{p - 1}} v(x) \leq C \quad \text{for } |x| > R_*.$$

Hence we can apply Theorem 1.2 to conclude that either v has fast decay at infinity, or there exist $C_1, C_2 > 0$ such that

$$(5.22) \quad C_1 \leq |x|^{\frac{2 + \tau}{p - 1}} v(x) \leq C_2 \quad \text{for all large } |x|.$$

Thus to complete the proof, we only have to show that (5.22) does not hold. Suppose that (5.22) holds, we will derive a contradiction.

Since v has finite Morse index over Ω , we may assume that v is stable in $\mathbb{R}^N \setminus B_R$.

Step 1. Suppose that $\tau > -2$, $p > 1$ and v is a stable positive solution of (1.1) in $\mathbb{R}^N \setminus B_R$ with $R > R_*$. Then there exists $R_0 > R$ such that for every $\gamma \in [1, 2p + 2\sqrt{p(p - 1)} - 1)$

and every $r > R_0$, we have

$$(5.23) \quad \int_{\{R_0 < |x| < r\}} |x|^l v^{\gamma+p} \leq C + Dr^{N' - \frac{(2+\tau)\gamma+2p+\tau}{p-1}},$$

where C and D are positive constants depending on m, p, N', τ, R, R_0 but not on r .

Since v is stable in $\mathbb{R}^N \setminus B_R$, Proposition 1.5 holds with $\Omega = \mathbb{R}^N \setminus B_R$. We now choose a suitable test function. We fix $\varphi_0 \in C^2(\mathbb{R})$ as in the proof of Theorem 5.1. Then define

$$\tilde{\xi}_r(x) = \begin{cases} 1 & \text{if } |x| \leq R_*/2, \\ 1 - \varphi_0(\frac{2|x|}{r}) & \text{if } |x| \geq R_*/2. \end{cases}$$

We may then prove (5.23) in the same way as in Step 1 of the proof of Theorem 5.1.

Step 2. Reaching a contradiction when (5.13) holds.

As in the proof of Theorem 5.1, under our assumption on p , we can find $\gamma_0 \in [1, \gamma(p))$ such that $\Delta(N', p, \gamma_0, \tau) = 0$, that is

$$N' - \frac{(2+\tau)\gamma_0 + 2p + \tau}{p-1} = 0.$$

Choosing $\gamma = \gamma_0$ in (5.23), we obtain

$$\int_{\{R_0 < |x| < r\}} |x|^l v^{\gamma_0+p} \leq C + D.$$

On the other hand, using (5.22) we deduce

$$\begin{aligned} \int_{\{R_0 < |x| < r\}} |x|^l v^{\gamma_0+p} &\geq C_1^{\gamma_0+p} \int_{\{R_0 < |x| < r\}} |x|^{l - \frac{2+\tau}{p-1}(\gamma_0+p)} = C_1^{p+\gamma_0} \int_{R_0}^r s^{-1} ds \\ &= C_1^{\gamma_0+p} \log(r/R_0) \rightarrow \infty \quad \text{as } r \rightarrow \infty, \end{aligned}$$

a contradiction. This completes our proof. \square

We next use the Kelvin transformation to show that the conclusions of both Theorems 5.1 and 5.2 continue to hold when

$$(5.24) \quad \tilde{p}_c(N', \tau) < p < \min \left\{ \frac{N' + 2 + 2\tau}{N' - 2}, p_c(N, 0) \right\};$$

and moreover, when $p \in \left(\frac{N'+\tau}{N'-2}, \tilde{p}_c(N', \tau) \right]$, V_∞ is a stable solution of (1.1) over $\mathbb{R}^N \setminus \{0\}$. Clearly Theorems 1.12 and 1.14 follow from these.

We only consider the case of Theorem 5.1, the proof for the case of Theorem 5.2 is analogous.

Theorem 5.3. *Let Ω_0 be a bounded domain in \mathbb{R}^N ($N \geq 2$) containing 0, and let v be a positive solution of (1.1) with $\Omega = \Omega_0 \setminus \{0\}$. If v has finite Morse index, then $x = 0$ must be a removable singularity of v provided that (5.24) holds.*

On the other hand, if $\frac{N'+\tau}{N'-2} < p \leq \tilde{p}_c(N', \tau)$, then problem (1.1) has a stable positive solution with an isolated singularity at 0.

Proof. Let p be in the range given by (5.24) and suppose that v is a positive solution of (1.1) with finite Morse index. Then there exists $R > 0$ such that v is stable in $B_R \setminus \{0\}$. Therefore, the function w given by

$$w(y) = |x|^{N'-2}v(x), \quad y = \frac{x}{|x|^2}$$

is a stable solution of

$$(5.25) \quad -\operatorname{div}(|y|^\theta \nabla w) = |y|^\beta w^p \quad \text{in } \mathbb{R}^N \setminus \overline{B_{1/R}},$$

with $\tau' = \tau'(p, \tau) := \beta - \theta = (N' - 2)(p - 1) - (4 + \tau) > -2$ (due to $p > \tilde{p}_c(N', \tau) > \frac{N'+\tau}{N'-2}$).

We now show that Theorem 5.2 can be used to conclude the proof. To this end, we need to analyze the function $f(p)$ when τ is replaced by τ' . To stress the dependence of $f(p)$ on τ , we write $f(p) = f_\tau(p)$. For (p, τ) given above, and $\tau' = \tau'(p, \tau) > -2$, we now consider the function $f_{\tau'}(q)$ for $q \in (1, \infty)$. From our analysis on $f_\tau(p)$ we know that

$$\begin{aligned} f_{\tau'}(q) &< \frac{(N' - 2)^2}{4} \quad \forall q \in (1, \tilde{p}_c(N', \tau')) \cup (p_c(N', \tau'), \infty), \\ f_{\tau'}(q) &> \frac{(N' - 2)^2}{4} \quad \forall q \in (\tilde{p}_c(N', \tau'), p_c(N', \tau')). \end{aligned}$$

A simple calculation shows that

$$p - \frac{N' + 2 + 2\tau}{N' - 2} = \frac{N' + 2 + 2\tau'}{N' - 2} - p \quad \text{and} \quad f_{\tau'}(p) = f_\tau(p).$$

Thus under our assumption on p , we have

$$p > \frac{N' + 2 + 2\tau'}{N' - 2} \quad \text{and} \quad f_{\tau'}(p) = f_\tau(p) > \frac{(N' - 2)^2}{4}.$$

By the property of the function $f_{\tau'}(q)$, the above inequalities imply that

$$p \in \left(\frac{N' + 2 + 2\tau'}{N' - 2}, p_c(N', \tau') \right).$$

In view of (5.24), we conclude that

$$\frac{N' + 2 + 2\tau'}{N' - 2} < p < \min\{p_c(N', \tau'), p_c(N, 0)\}.$$

Therefore Theorem 5.2 applies to (5.25), and $w(y)$ has fast decay at ∞ . This implies that $x = 0$ is a removable singularity of v .

Finally we show that V_∞ is stable in $\mathbb{R}^N \setminus \{0\}$ when $\frac{N'+\tau}{N'-2} < p \leq \tilde{p}_c(N', \tau)$. This is equivalent to showing that (3.4) holds for such p , which would follow if

$$\frac{(N' - 2)^2}{4} - pC_0^{p-1} = \frac{(N' - 2)^2}{4} - f(p) \geq 0.$$

But for $p \in (\frac{N'+\tau}{N'-2}, \tilde{p}_c(N', \tau)]$ we do have $f(p) \leq \frac{(N'-2)^2}{4}$. Thus V_∞ is indeed stable for such p . \square

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